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AD

PROPAGATION OF A CIRCULAR CRACK IN TORSION

May 1976

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BENET WEAPONS LABORATORY
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INTRODUCTION

This report deals with the mathematical aspects of a problem of importance in the failure of structures, namely, the sudden spreading of a crack in a stressed medium. A previous study on the dynamic problem of torsion of a circular crack was made by Sih and Embley [1]. Their work, however, is restricted to the case of a crack which opens instantaneously to a fixed radius and thereafter remains stationary. Most work on crack propagation under other loading conditions is based on the mathematically expedient supposition that propagation occurs at constant speed. It is more realistic to consider the speed of propagation as a function of time based on some physical hypothesis. Recently, Kostrov [2] has proposed an analytical technique for treating a variable speed crack and has obtained results for the case of anti-plane loading conditions. In this report Kostrov's technique is applied to the case of torsional loading conditions.

Kostrov's method has the distinct advantage over other methods, notably the Weiner-Hopf method and the dual integral equation method, in that mixed problems with time dependent boundaries can be treated. Moreover, the near-field solution can be deduced rather easily, at least for torsion and anti-plane strain conditions. An inherent drawback of Kostrov's method is that for a finite length crack the

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[1] G. C. SIH and G. T. EMBLEY (1972) Journal of Applied Mechanics, 395-400. Sudden Twisting of a Penny-Shaped Crack.

[2] B. V. KOSTROV (1966) Journal of Applied Mathematics and Mechanics, 30(6), 1241-1248. Unsteady Propagation of Longitudinal Shear Cracks.

analytical results are limited to the time interval for which waves enamating from opposite points on the crack border have not yet come into contact; it appears that numerical integration is required for the time in which wave interaction occurs. On the basis of the results of the study by Sih and Embley [1], however, the dynamic stress intensity factor reaches its maximum value before wave interaction occurs. Thus it is reasonable to suppose that the results obtained in this report on the dynamic stress intensity factor represent the maximum value that can be expected.

PROBLEM FORMULATION AND SOLUTION

Consider an infinite elastic solid with shear modulus μ and density ρ referred to cylindrical coordinates r,θ,z. At time $t=o$ a circular crack of radius a occupies the plane $z=o$; for time $t>o$ the crack radius is $r^*(t) = a + R(t)$, where $R(t)$ is a monotonic increasing function, with $R(o)=o$. It is assumed that $\frac{dR}{dt}<c$, where $c=(\mu/\rho)^{1/2}$ is the torsional wave speed of the solid. Suppose that the solid is loaded by tractions of equal magnitude applied to the adjacent edges of the crack surface, and that these tractions are independent of θ. Then the displacement $v(r,z,t)$ will be anti-symmetric in z, so that it is sufficient to consider only the region $z\geq o$. For this problem the stress components are

$$\sigma_{\theta z} = \mu v_{,z} \quad , \quad \sigma_{\theta r} = \mu(v_{,r} - \frac{1}{r}v) \quad , \tag{1}$$

[1] G. C. SIH and G. T. EMBLEY (1972) Journal of Applied Mechanics, 395-400. Sudden Twisting of a Penny-Shaped Crack.

and the displacement satisfies the equation

$$v_{,rr} + \frac{1}{r} v_{,r} - \frac{1}{r^2} v + v_{,zz} = v_{,tt} \quad , \tag{2}$$

wherein for convenience ct has been replaced by t. The boundary conditions on z=o may be written as

$$\sigma_{\theta z} = f(r,t) \quad , \quad r<r^*(t) \quad , \tag{3}$$

$$v = o \quad , \; r>r^*(t) \quad , \tag{4}$$

where f(r,t) is a continuous function. Since by superposition any initial displacements may be separated, then without loss in generality let

$$v = o, \; v_{,t} = o \quad , \quad t = o \; . \tag{5}$$

The problem as formulated by equations (1)-(5) constitutes a mixed boundary value problem. In order to apply Kostrov's method it is first necessary to obtain a relationship between the displacement and the stress $\sigma_{\theta z}$ on the plane z=o. To this end let

$$v(\xi,z,p) = \int_0^\infty e^{-pt} dt \int_0^\infty rJ_1(\xi r)\, v(r,z,t)\; dr \quad , \tag{6}$$

$$\Sigma(\xi,z,p) = \int_0^\infty e^{-pt} dt \int_0^\infty rJ_1(\xi r)\, \sigma(r,z,t)\; dr \quad , \tag{7}$$

denote Hankel-Laplace transforms; in (7) σ denotes $\sigma_{\theta z}$. Applying (6) to equation (2) and using (5) gives

$$v_{,zz} - \alpha^2 v = o \quad , \quad z \geqslant o \quad , \tag{8}$$

where $\alpha^2 = \xi^2 + p^2$. The appropriate solution of (8) is

$$v(\xi,z,p) = v(\xi,p)\, e^{-\alpha z} \quad , \tag{9}$$

provided that $Re\{\alpha\} \geq o$. Likewise, applying (7) to (1) and using (9) gives

$$\Sigma(\xi,z,p) = \Sigma(\xi,p)e^{-\alpha z} \quad , \tag{10}$$

and

$$\Sigma(\xi,p) = -\mu\alpha v(\xi,p) \quad . \tag{11}$$

Formula (11) gives the relationship between the transformed stress and displacement components on $z=o$. Now substituting (11) into (9) and applying the Hankel-Laplace inversion formula gives

$$\mu v(r,z,t) = -\frac{1}{2\pi i}\int_o^\infty \xi J_1(\xi r)\, d\xi \int_B e^{pt}\,\frac{\Sigma(\xi,p)}{\alpha}\, e^{-\alpha z}\, dp \quad , \tag{12}$$

where B denotes the well-known Bromwich contour. Using the result

$$\frac{1}{2\pi i}\int_B e^{pt}\,\frac{e^{-\alpha z}}{\alpha}\, dp = \begin{cases} o \;, & o<t<z \;, \\ J_o(\xi u) \;, & z<t<\infty \;, \end{cases} \tag{13}$$

where $u = (t^2-z^2)^{1/2}$, together with the convolution theorem gives

$$\mu v(r,z,t) = -\int_o^{t-z} dt' \int_o^\infty r'\sigma(r',t')\, dr' \int_o^\infty \xi J_1(r\xi)J_1(r'\xi)J_o(s\xi)d\xi, \tag{14}$$

wherein $s = [(t-t')^2-z^2]^{1/2}$. The inner integral may be expressed as [3]

$$\int_o^\infty \xi J_1(r\xi)J_1(r'\xi)J_o(s\xi)\, d\xi = \begin{cases} o \;, & o<r'<|s-r| \;, \\ \dfrac{1}{\pi r r'}\cot\phi \;, & |s-r|<r'<s+r \;, \\ o \;, & r+s<r' \;, \end{cases} \tag{15}$$

[3] I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

where

$$\cot\phi = (r^2+r'^2-s^2)\,\{[s^2-(r-r')^2]\,[(r+r')^2-s^2]\}^{-1/2} \quad . \tag{16}$$

Hence (12) becomes

$$\mu v(r,z,t) = -\frac{1}{\pi r}\int_R\!\!\int \sigma(r',t')\cot\phi\;\;dr'dt' \quad . \tag{17}$$

Region R is depicted in figures (1) and (2) for the situations in which $o<t<(r^2+z^2)^{1/2}$ and $(r^2+z^2)^{1/2}<t$, respectively. In figure (2) region R is bounded by the line $t'=o$ and the two hyperbolas $s=r+r'$ and $s=|r-r'|$. The dashed lines represent the limiting boundary of region R as $z\to o$, and in the sequel will be referred to as region R_o.

Formula (17) may be used in conjunction with Kostrov's method to determine the stress ahead of the crack perimeter. Thus putting $z=o$ in (17) and invoking conditions (3) and (4) gives

$$\int_{R_{o2}}\!\!\int \sigma(r',t')\cot\phi\;dr'dt' = -\int_{R_{o1}}\!\!\int f(r',t')\cot\phi\;dr'dt',\;\; r>r^*(t) \tag{18}$$

where $\phi = \phi/_{z=o}$, and where, as shown in figure (3) regions R_{o1} and R_{o2} denote portions of region R_o to the left and right, respectively, of the crack perimeter $r' = r^*(t')$. It is clear that $\sigma(r',t')=o$ in region R_{o3} .

Equation (18) is now recast in terms of the coordinates $\xi' = r'-t'$ and $\eta' = r'+t'$. Thus using the notation $\sigma(r',t') = \tau(\xi',\eta')$, $f(r',t') = g(\xi',\eta')$ gives

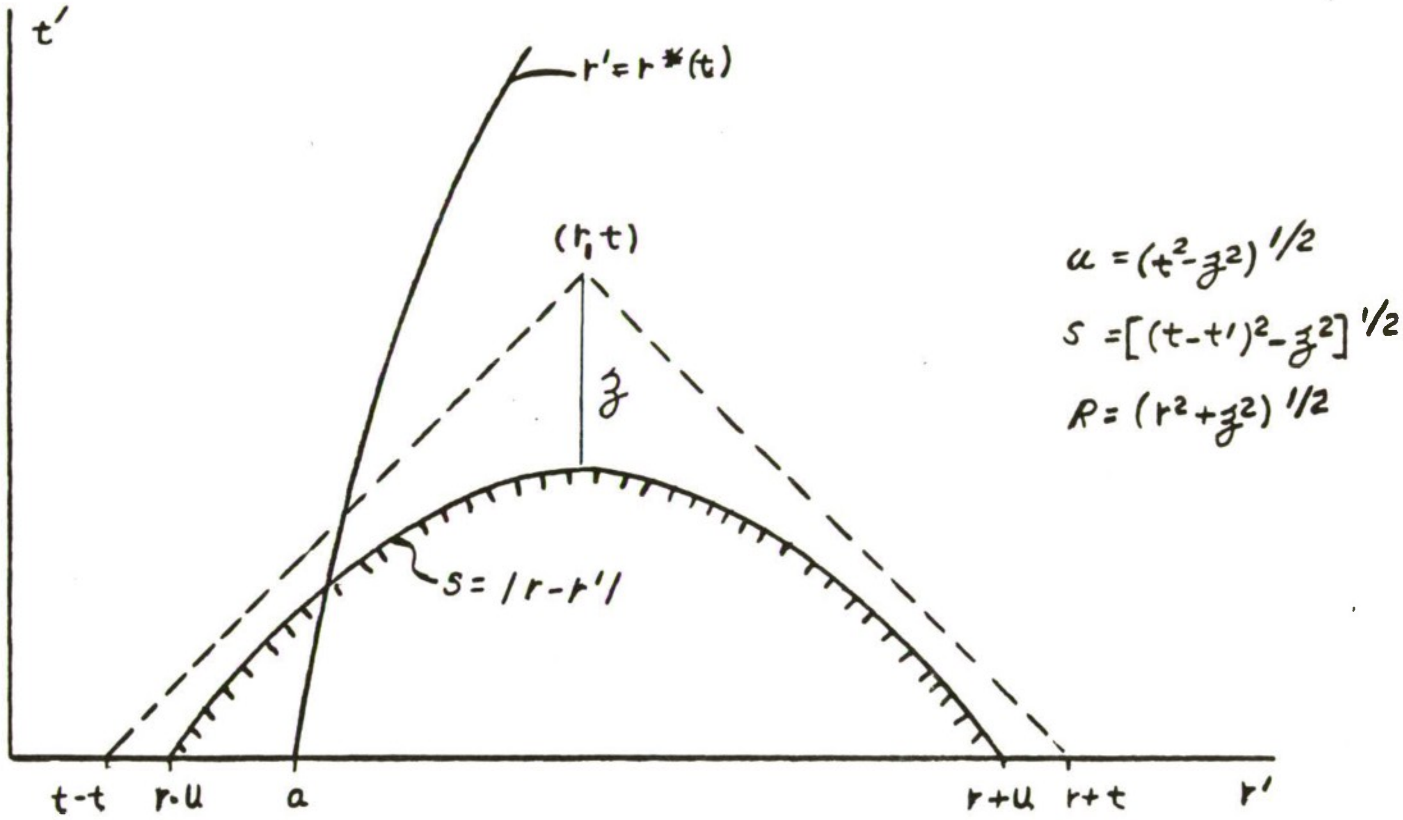

Figure 1. Solution Region: Drawn for the Case t<R, r>r*(t)

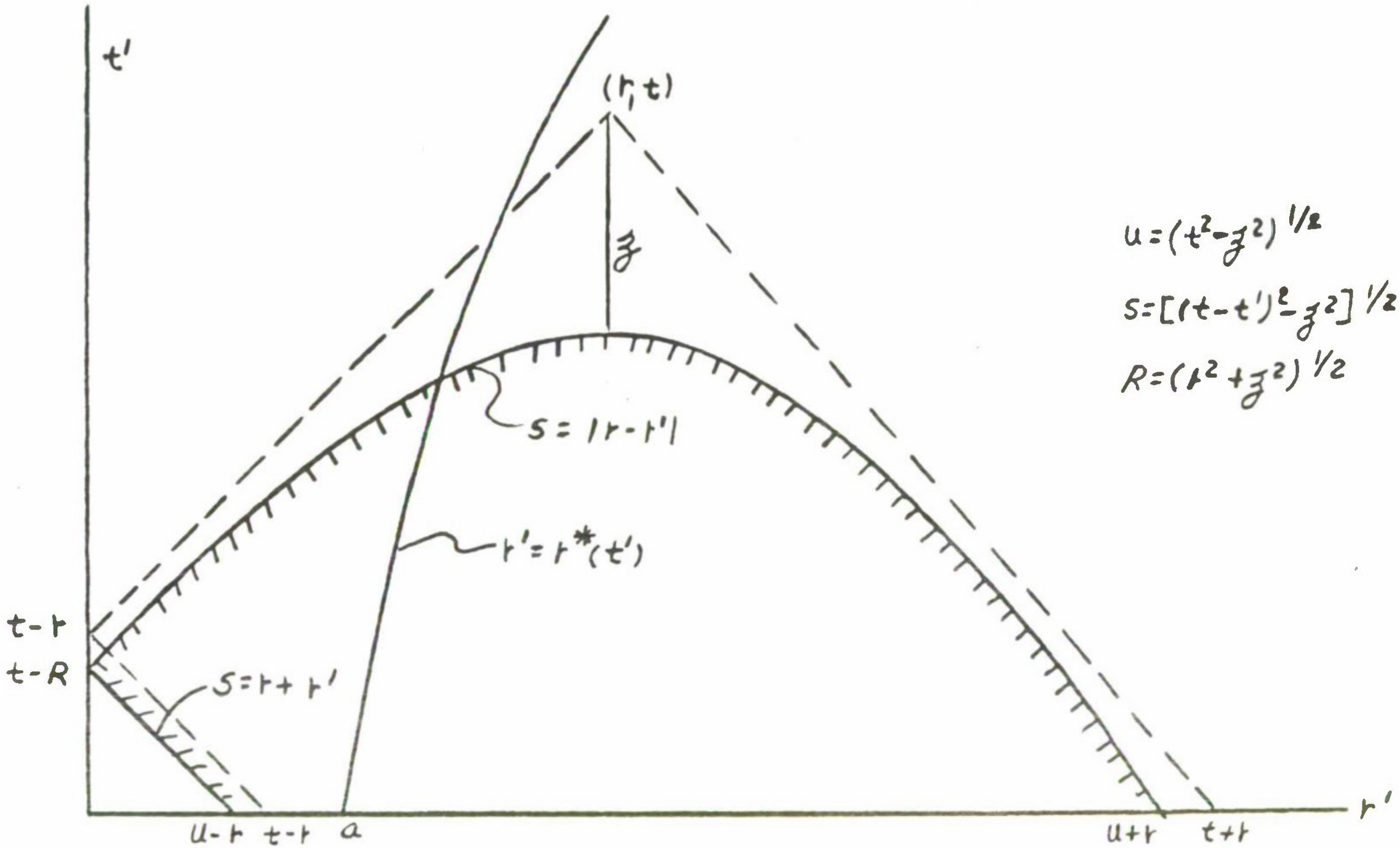

Figure 2. Solution Region: Drawn for the Case t>R, r>r*(t)

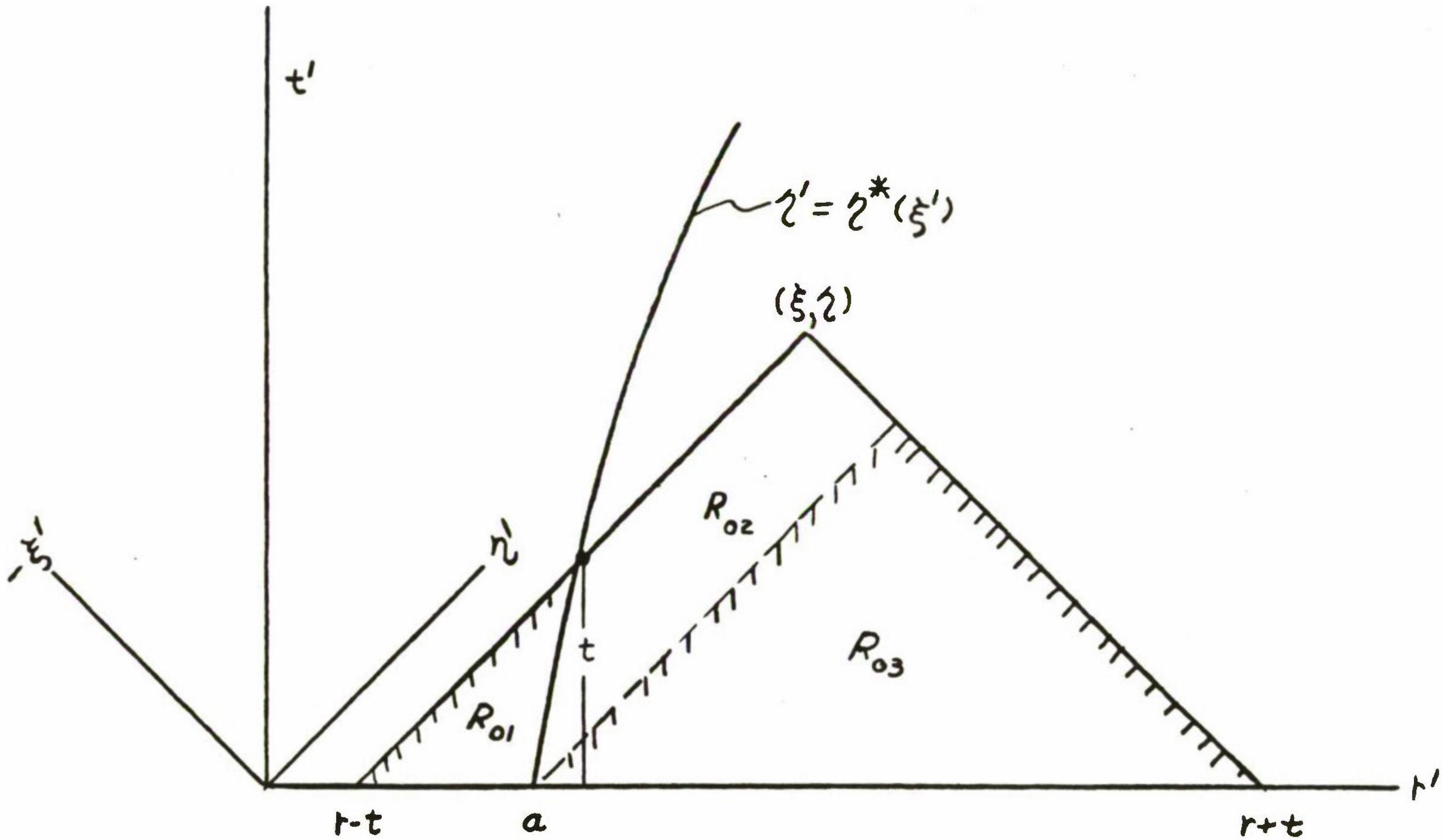

Figure 3. Diagram for Crack Plane Stress Solution

$$\int_{\xi}^{a} \frac{d\xi'}{(\xi'-\xi)^{1/2}(\eta+\xi')^{1/2}} \left\{ \int_{\eta^*(\xi')}^{\eta} \tau(\xi',\eta') \frac{N(\xi',\xi,\eta',\eta)}{(\eta-\eta')^{1/2}(\xi+\xi')^{1/2}} d\eta' + F(\xi',\xi,\eta) \right\} = 0, \tag{19}$$

where

$$F(\xi',\xi,\eta) = \int_{\xi'}^{\eta^*(\xi')} g(\xi',\eta') \frac{N(\xi',\xi,\eta',\eta)}{(\eta-\eta')^{1/2}(\xi+\eta')^{1/2}} d\eta' , \tag{20}$$

$$N(\xi',\xi,\eta',\eta) = 2\ (\xi'\eta'+\xi\eta) + (\eta'-\xi')\ (\eta-\xi) , \tag{21}$$

and $\eta'=\eta^*(\xi')$ denotes the crack perimeter. Note that since $r'=r^*(t')$ then $t'=r^{*-1}(r') = t^*(r')$, so that $\eta^*(\xi')$ is given implicitly by the formula

$$\eta^* = \xi' + 2\ t^* \left[\frac{1}{2}(\eta^*+\xi') \right] . \tag{22}$$

Now equation (19) is satisfied if the expression within braces vanishes. Moreover substituting

$$\tau(\xi',\eta') = (\xi+\eta')^{1/2}\ \psi(\xi',\xi,\eta') , \tag{23}$$

leads to an Abel type integral equation for the stress ahead of the crack perimeter of the form

$$\int_{\eta^*(\xi')}^{\eta} \psi(\xi',\xi,\eta') \left[(\eta-\eta')^{1/2}\ (2\xi-\xi'+\eta') + \frac{(\xi+\eta')(\xi'+\eta')}{(\eta-\eta')^{1/2}} \right] d\eta' = -F(\xi',\xi,\eta) . \tag{24}$$

The solution of (24) may be obtained in a straightforward way by use of Laplace transforms; the result is written as

$$\frac{d}{d\eta} [(\xi'+\eta)^2(\xi+\eta)^{1/2}\ \psi(\xi',\xi,\eta)] = -\frac{1}{\pi} \frac{\eta+\xi'}{(\eta+\xi)^{1/2}} \frac{d^2}{d\eta^2} \int_{\eta^*(\xi')}^{\eta} \frac{F(\xi',\xi,\eta')}{(\eta-\eta')^{1/2}} d\eta' . \tag{25}$$

To get the solution at the point (ξ,η) let $\xi'=\xi$ and substitute from (23). This gives

$$\frac{d}{d\eta}\left[(\eta+\xi)^2\,\tau(\xi,\eta)\right] = -\frac{1}{\pi}(\eta+\xi)^{1/2}\frac{d^2}{d\eta^2}\int_{\eta^*(\xi)}^{\eta}\frac{F(\xi,\xi,\eta')}{(\eta-\eta')^{1/2}}\,d\eta' . \qquad (26)$$

Now since $F(\xi,\xi,\eta') = (\xi+\eta')\,E(\xi,\eta')$, where

$$E(\xi,\eta') = \int_{\xi}^{\eta^*(\xi)} g(\xi,v)\,\frac{(\xi+v)^{1/2}}{(\eta'-v)^{1/2}}\,dv , \qquad (27)$$

then (26) may be written as

$$\frac{d}{d\eta}\left[(\eta+\xi)^2\tau(\xi,\eta)\right] = -\frac{1}{\pi}\frac{d}{d\eta}\left[(\eta+\xi)^{3/2}\frac{d}{d\eta}\int_{\eta^*(\xi)}^{\eta}\frac{E(\xi,\eta')}{(\eta-\eta')^{1/2}}\,d\eta'\right].\qquad (28)$$

Finally, substituting from (27) and performing one integration yields the result

$$\tau(\xi,\eta) = -\frac{1}{\pi(\eta+\xi)^{1/2}[\eta-\eta^*(\xi)]^{1/2}}\int_{\xi}^{\eta^*(\xi)} g(\xi,\eta')$$

$$\frac{(\xi+\eta')(\eta^*-\eta')^{1/2}}{\eta-\eta'}\,d\eta' , \quad \eta>\eta^*(\xi) . \qquad (29)$$

In terms of the original variables (29) becomes

$$\sigma(r,t) = -\frac{1}{\pi r^{1/2}[r-r^*(\tau)]^{1/2}}\int_{|r-t|}^{r^*(\tau)} f(r',t-r+r')$$

$$\frac{r'^{1/2}[r^*(\tau)-r']^{1/2}}{r-r'}\,dr' , \qquad r>r^*(t) . \qquad (30)$$

In formula (30) the quantity τ, (not to be confused with the stress $\tau(\xi,\eta)$), is given implicitly by the relationship $t-\tau = r-r^*(\tau)$, as may be seen from figure (3). The quantity τ represents the retarded time associated with the moving crack border, that is, a point r experiences

at time t the disturbance, in the form of a wave, caused by the moving crack border when the border location was at $r^*(\tau)$.

The result (30) is valid for the time interval $r-a<t<r+a$. This interval corresponds to the time when the disturbance from the moving crack perimeter point $(r^*(t), \theta+\pi)$ has not yet reached the crack position $(r^*(t),\theta)$. To determine $\sigma(r,t)$ for larger values of time it appears necessary to replace $f(r',t')$ by $-\sigma(r',t')$ on those portions of the range of integration on which the stress is unknown. This procedure corresponds to repeated diffraction of the waves at the crack perimeter. It does not appear feasible to carry out this procedure analytically since multiple integrals are involved.

Since it is clear from figure (3) that the condition $r>r^*(t)$ is equivalent to the condition $r>r^*(\tau)$, it follows that formula (30) defines the stress ahead of the actual crack tip. Moreover, formula (30) gives a singularity of the order 1/2 at the actual crack tip location. That this is so follows from a Lagrange expansion of the quantity $r-r^*(\tau)$, which is of the form [4]

$$r-r^*(\tau) = r-r^*(t) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial t}\right)^{n-1} \dot{r}^*(t)\, \rho^n(t) \ , \qquad (31)$$

[4]E. T. WHITTAKER and G. N. WATSON (1927) Modern Analysis, Fourth Edition, Cambridge University Press.

where

$$\dot{r}^*(t) = \frac{dr^*}{dt}\ , \text{ and } \rho(t) = \{[r-r^*(t)]^2 + z^2\}^{1/2}\ .$$

On z=o (31) gives

$$r-r^*(\tau) = \frac{r-r^*(t)}{1-\dot{r}^*(t)} + \text{terms of order } \rho^2\ . \tag{32}$$

Hence as $r \to r^*(t)$ the term $[r-r^*(\tau)]^{-1/2}$ in formula (30) may be replaced by the term $\left[\frac{r-r^*(t)}{1-\dot{r}^*(t)}\right]^{-1/2}$, so that in the neighborhood of the crack perimeter

$$\sigma(r,t) \approx -\frac{k}{\pi\sqrt{r[r-r^*(t)]}}\ , \tag{33}$$

where k is the coefficient of stress intensity

$$k = \frac{\sqrt{1-\dot{r}^*(t)}}{\sqrt{r^*(t)}} \int_{|r^*(t)-t|}^{r^*(t)} f(r',t-r^*(t)+r')\ \frac{\sqrt{r'}}{\sqrt{r^*(t)-r'}}\ dr'\ . \tag{34}$$

With the crack plane stress completely determined (at least for the time interval previously noted), it is now a relatively simple matter to obtain the crack displacement. Thus reverting to the coordinates $\xi=r-t$, $\eta=r+t$, letting $w(\xi,\eta) = v(r,t)$, and employing the previous designation for the stress component in formula (17) gives for z=o

$$-2\pi(\xi+\eta)\ w(\xi,\eta) = \int_{\xi}^{\xi^*(\eta)} \frac{d\xi'}{(\xi'-\xi)^{1/2}(\eta+\xi')^{1/2}}\ G(\xi',\xi,\eta)$$

$$+\int_{\xi^*(\eta)}^{a} \frac{d\xi'}{(\xi'-\xi)^{1/2}(\eta+\xi')^{1/2}} \left[F(\xi',\xi,\eta) + \int_{\eta^*}^{\eta} \tau(\xi',\eta') \frac{N(\xi,'\xi,\eta',\eta)\ d\eta'}{(\eta-\eta')^{1/2}(\xi+\eta')^{1/2}} \right] \tag{35}$$

where

$$G(\xi',\xi,\eta) = \int_{\xi'}^{\eta} g(\xi',\eta') \frac{N(\xi',\xi,\eta',\eta)}{(\eta-\eta')^{1/2}(\xi+\eta')^{1/2}}\ d\eta' \ . \tag{36}$$

In (35) $\xi^*(\eta)$ denotes the ξ' co-ordinate of the intersection of the line $\eta'=\eta$ and the crack locus $\eta'=\eta^*(\xi')$, i.e. $\xi^*(\eta) = \eta^{*-1}_{(\eta)}$. Now from (19) the bracketed quantity within the second integral vanishes. Hence rewriting (35) in terms of the original variables yields

$$\mu v(r,t) = -\frac{1}{\pi r}\int\int_{R_1} f(r',t')\cot\phi_0\ dr'dt' \ ,\ r<r^*(t), \tag{37}$$

where region R_1 is shown in figure (4). Formula (37) represents the displacement for general dynamic loading of a circular crack of initial radius a. It is of interest to note that the solution is independent of the stress ahead of the crack perimeter. Expressions (30) and (37) will now be utilized to study results for two particular cases.

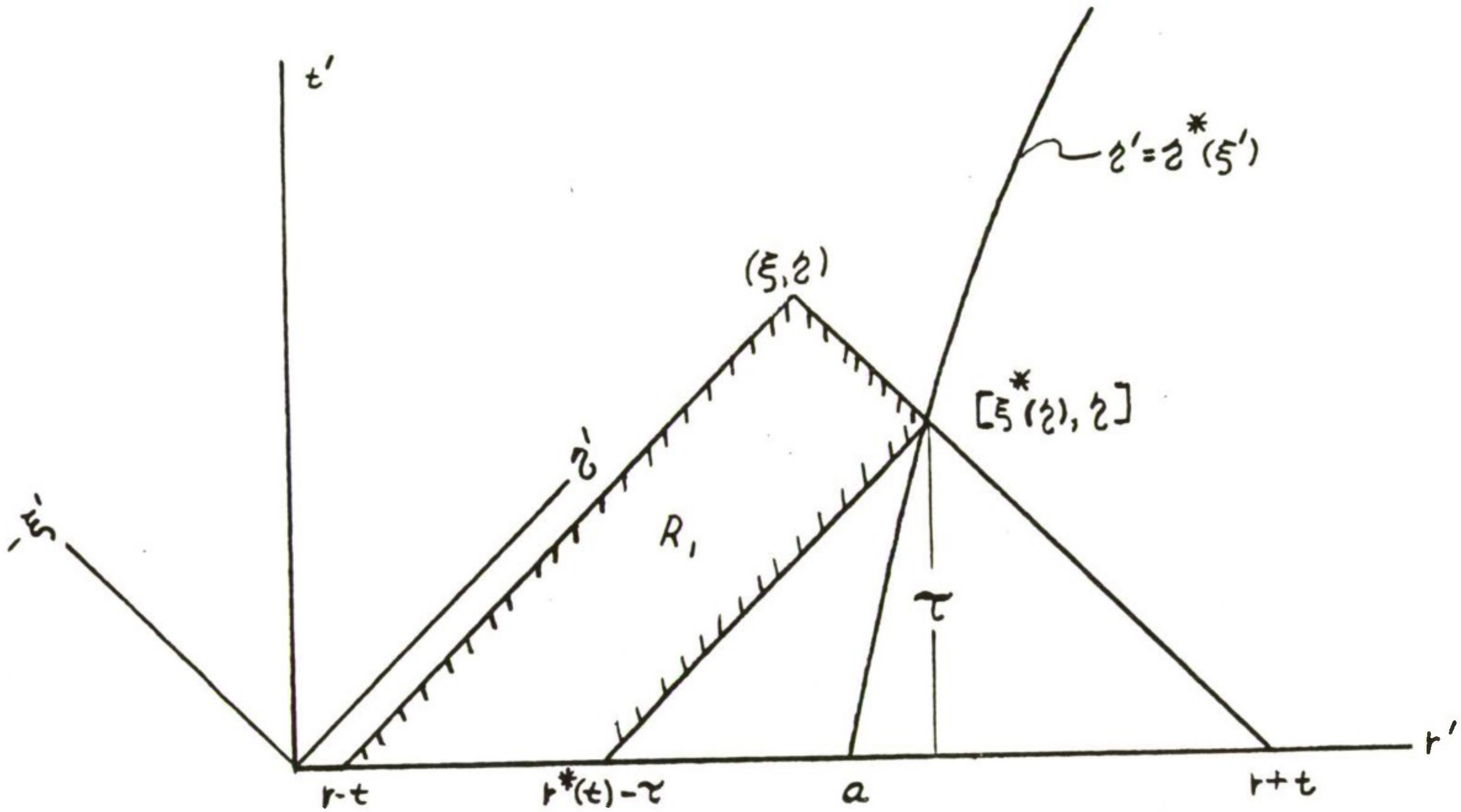

Figure 4. Diagram for Crack Plane Displacement Solution

Case 1. Initial Value Problem.

In formula (30) put

$$f(r,t) = o \quad , \quad o<r<a$$
$$= -\overset{(o)}{\sigma}(r) \quad , \quad a<r<r^*(t) \quad , \tag{38}$$

where $\overset{(o)}{\sigma}(r)$ is some initial state of stress. The resulting expression represents the perturbed stress arising from the motion of the crack perimeter. As an illustration let

$$\overset{(o)}{\sigma}(r) = \frac{S}{2r\sqrt{r^2-a^2}} (2r^2-a^2)\, H(r-a) \quad . \tag{39}$$

Expression (39) is the result given by Sneddon [5] for the stress on the crack plane z=o under conditions of uniform stress S at infinity. Introducing (39) into (30) yields

$$\sigma(r,t) = \frac{S}{2\pi\sqrt{r[r-r^*(\tau)]}} \int_a^{r^*(\tau)} \frac{2r'^2-a^2}{\sqrt{r'}(r'^2-a^2)} \frac{\sqrt{r^*(\tau)-r'}}{r-r'} dr', \ r>r^*(t) \ . \tag{40}$$

Since the initial stress on the freshly formed portion of the crack surface has been annulled, then, by superposition, (40) combined with (39) gives the actual stress ahead of the crack perimeter for the case of uniform stress S at infinity. The integral may be expressed in terms of known elliptic integrals; these expressions are given in the appendix.

In the limit as $r \to r^*(t)$ it is found that the stress intensity

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[5] I. N. SNEDDON and M. LOWENGRUB (1969) Crack Problems in the Classical Theory of Elasticity, Wiley and Sons.

factor for this case is

$$k = -\frac{S}{2}\left[\frac{1-\dot{r}^*(t)}{r^*(t)}\right]^{1/2} \int_a^{r^*(t)} \frac{2r'^2-a^2)\ \ dr'}{\sqrt{r'(r'^2-a^2)[r^*(t)-r']}} \quad . \tag{41}$$

Explicit expressions for the integral term may be obtained from those given in the appendix by putting $r=r^*(\tau) = r^*(t)$.

The perturbed crack displacement for this case follows by substituting (38) into (37). Reference to figure (4) shows that for $o<r-t<a$ the result may be expressed as

$$\mu v(r,t) = \frac{1}{\pi r} \int_a^{r^*(\tau)} \overset{(o)}{\sigma}(r')dr' \int_{\tau-r^*(\tau)+r'}^{t+r-r'} \cot\phi\ dt' \quad , \tag{42}$$

wherein $\overset{(o)}{\sigma}(r')$ is given by (39) and τ is given implicitly by $t-\tau = r^*(\tau)-r$. With the aid of tables [3] it is found that (42) may be written as

$$\mu v(r,t) = \frac{1}{\pi r} \int_a^{r^*(\tau)} \overset{(o)}{\sigma}(r')\ \ dr'\ \ I(r',r,t) \quad , \tag{43}$$

where

$$I(r',r,t) = \frac{r^2+r'^2}{r+r'} F(\kappa,q) - (r+r')\,E(\kappa,q)$$

$$+ \frac{1}{w}\{[(r+r')^2-w^2]\,[w^2-(r-r')\ ]\}^{1/2} \quad ,$$

in which $F(\kappa,q)$ and $E(\kappa,q)$ are elliptic integrals of the first and second kind, respectively, with

$$\kappa = \sin^{-1}\left\{\frac{r+r'}{w}\left[\frac{w^2-(r'-r)^2}{4rr'}\right]^{1/2}\right\} \quad , \quad q = \frac{(4rr')^{1/2}}{r+r'}$$

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[3] I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

and $w = t-\tau+r^*(\tau)-r'$. For $o<t-r<a$ identical results are found, hence (43) is valid for $o<|t-r|<a$.

Case 2: Suddenly Appearing Crack

For this case let $f(r,t) = -S$. Then formula (30) gives

$$\sigma(r,t) = \frac{S}{\pi\sqrt{r[r-r^*(\tau)]}} \int_{|r-t|}^{r^*(\tau)} \frac{\sqrt{r'[r^*(\tau)-r']}}{r-r'}\, dr' \quad r>r^*(t). \tag{44}$$

The result (44), superposed with a constant stress S, gives the solution for a suddenly appearing crack under uniform stress S at infinity. Here it is assumed that the crack opens instantaneously to radius a and thereafter is located at $r=r^*(t)$. For this case the integral may be evaluated explicitly; the result is

$$\begin{aligned}\sigma(r,t) = \frac{S}{\pi\sqrt{r[r-r^*(\tau)]}} \Bigg\{ & \sqrt{|t-r|\,[r^*(\tau)-|t-r|]} \\ & + \frac{1}{2}[2r-r^*(\tau)] \sin^{-1}\left[\frac{2}{r^*(\tau)}\sqrt{|t-r|\,[r^*(\tau)-|t-r|]}\right] \\ & - \sqrt{r[r-r^*(\tau)]} \sin^{-1}\left[\frac{2\sqrt{r|t-r|[r^*(\tau)-|t-r|][r-r^*(\tau)]}}{r^*(\tau)\,(r-|t-r|)}\right] \Bigg\}\end{aligned} \tag{45}$$

As an example, if the crack speed is a linear function of time, say $r^*(t) = a+\beta t$, with $\beta<1$, then since $t-\tau = r-r^*(\tau)$ it follows that $\tau = \frac{t-r+a}{1-\beta}$. Putting this result in (45) gives for $r-r^*(\tau)<t<r$

$$\sigma(r,t) = \frac{S}{\pi\sqrt{r(r-a-\beta t)}} \Big\{ \sqrt{(r-t)(t-r+a)} + \frac{1}{2}\,\frac{(2-\beta)r-a-\beta t}{\sqrt{1-\beta}}$$

$$\sin^{-1}\left[\frac{2\sqrt{1-\beta}}{a+\beta(t-r)}\sqrt{(r-t)(t-r+a)}\right] - \frac{\sqrt{r(r-a-\beta t)}}{\sqrt{1-\beta}}$$

$$\sin^{-1}\left[\frac{2}{t[a+\beta(t-r)]}\sqrt{r(r-t)(t-r+a)(r-a-\beta t)}\right] \Big\} \qquad (46)$$

The crack surface displacement may likewise be obtained explicitly. Thus substituting into (37) and referring to figure (4) gives for

$o<r-t<a$

$$\mu v(r,t) = \frac{S}{\pi r}\int_0^t dt' \int_{r+t'-t}^{r^*(\tau)-|t'-\tau|} \cot\phi \, dr' \quad , \quad r<r^*(t) \quad . \qquad (47)$$

Then using tables [3] gives

$$\mu v(r,t) \doteq \frac{S}{\pi r}\int_0^t J(r,s,t)\; ds \quad , \quad r<r^*(t) \quad , \qquad (48)$$

where

$$J(r,s,t) = (r-s)F(\kappa,q) + (r+s)E(\kappa,q) - \frac{1}{\lambda}\sqrt{[(r+s)^2-\lambda^2][\lambda^2-r-s)^2]} \; ,$$

in which

$$\kappa = \sin^{-1}\frac{r+s}{\lambda}\,\frac{\sqrt{\lambda^2-(r-s)^2}}{\sqrt{4rs}} \quad , \quad q = \frac{\sqrt{4rs}}{r+s}$$

and $\lambda = r^*(\tau) - |t-s-\tau|$.

It is of interest to note that if $r-a<t<r+a$ the results simplify

[3] I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

even further, since for this time interval a point on the crack surface has not yet experienced the effect of the waves from the crack perimeter. For this case formula (48) is replaced by

$$\mu v(r,t) = \frac{S}{\pi r} \int_0^t ds \int_{r-s}^{r+s} \cot \phi \, dr' \quad , \tag{49}$$

which, upon evaluation of the inner integral, gives

$$\mu v(r,t) = \frac{S}{\pi r} \int_0^t [(r-s)K(\frac{\sqrt{4rs}}{r+s}) + (r+s)E(\frac{\sqrt{4rs}}{r+s})] \, ds, \tag{50}$$

where K and E are complete elliptic integrals. Now substituting s=rw and applying the transformation formulae

$$K(\frac{2\sqrt{w}}{1+w}) = (1+w)K(w)$$

$$E(\frac{2\sqrt{w}}{1+w}) = \frac{2}{1+w} E(w) - (1-w)K(w)$$

gives

$$v(r,t) = \frac{2Sr}{\pi} \int_0^{t/r} E(w) \, dw \quad , \quad r-a<t<r+a \quad . \tag{51}$$

The final result follows by using

$$\int_0^{t/r} E(w) \, dw = \frac{\pi}{2} \frac{t}{r} \left[1 + \sum_{n=1}^{\infty} \frac{[(2n)!]^2 (t/r)^{2n}}{(2n+1)2^{4n}(n!)^4} \right] \tag{52}$$

The resulting expression is equivalent to that which would be obtained on the surface of a half-space loaded by a step function load of magnitude S.

DISCUSSION

The analytical results contained herein may be used as a basis for studying the physical aspects of the dynamic torsional fracturing process. As demonstrated by Kostrov [2] in the case of anti-plane strain, the crack propagation speed may be made definite by applying the physical hypothesis that the work done in the rupture process depends only on the crack speed. This hypothesis, together with Barenblatt's cohesion model leads to a differential equation governing the crack perimeter locus.

Another useful extension of the results of this report is to incorporate the effects of a plastic zone near the crack border. This may be done by using Dugdale's model as shown by Achenbach [6] for the case of anti-plane strain. These considerations, it is felt, are more appropriately left as subjects of future investigations.

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[2] B. V. KOSTROV (1966) Journal of Applied Mathematics and Mechanics, 30(6), 1241-1248. Unsteady Propagation of Longitudinal Shear Cracks.

[6] J. D. ACHENBACH (1970) International Journal of Engineering Science, Vol 8, 947-966. Brittle and Ductile Extension of a Finite Crack by a Horizontally Polarized Shear Wave.

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6. J. D. ACHENBACH (1970) International Journal of Engineering Science, Vol 8, 947-966. Brittle and Ductile Extension of a Finite Crack by a Horizontally Polarized Shear Wave.

APPENDIX

The integral term in equation (40) may be reduced as follows:

Let

$$A = \int_a^b \frac{(2x^2-a^2)}{\sqrt{x(x^2-a^2)}} \; \frac{\sqrt{b-x}}{r-x} \, dx \quad , \quad r \; b \; , \tag{A1}$$

where $b = r^*(\tau)$. Changing the integration variable to $u = r-x$ leads to

$$A = -a^2 I + 2J \quad , \tag{A2}$$

where

$$I = I_1 - \beta I_2 \quad , \tag{A3}$$

$$J = J_1 - \beta J_2 \quad , \tag{A4}$$

$$I_1 = \int_\beta^\alpha \frac{du}{\sqrt{(r-u)(\alpha-u)(u-\beta)(c-u)}} \tag{A5}$$

$$I_2 = \int_\beta^\alpha \frac{du}{u\sqrt{(r-u)(\;-u)(u-\;)(c-u)}} \tag{A6}$$

$$J_1 = \int_\beta^\alpha \frac{\sqrt{r-u}^3 \, du}{\sqrt{(\alpha-u)(c-u)(u-\beta)}} \quad , \tag{A7}$$

$$J_2 = \int_\beta^\alpha \frac{\sqrt{r-u}^3 \; du.}{u\sqrt{(\alpha-u)(c-u)(u-\beta)}} \quad . \tag{A8}$$

In (A3) - (A8), $\beta=r-a$, $\alpha=r-b$, $c=r+a$, so that $c>r>\alpha>\beta$. Now from tables [3] it is found that

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[3] I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

$$I_1 = \frac{2}{\sqrt{(c-\alpha)(r-\beta)}} \; K\left(\frac{\sqrt{(c-r)(\alpha-\beta)}}{\sqrt{(c-\alpha)(r-\beta)}}\right) \quad , \tag{A9}$$

where K is the complete elliptic integral of the 1st kind. Hence replacing the original variables gives

$$I_1 = \frac{2}{\sqrt{2ar^*(\tau)}} \; K\left(\frac{\sqrt{r^*(\tau)-a}}{\sqrt{2r^*(\tau)}}\right) \quad . \tag{A10}$$

$$I_2 = \frac{2[a+r^*(\tau)]}{(r+a)[r-r^*(\tau)]\sqrt{2ar^*(\tau)}} \; II\left(\pi/2, \frac{(r+a)[a-r^*(\tau)]}{2a[r-r^*(\tau)]} \; , \; \frac{\sqrt{r^*(\tau)-a}}{2r^*(\tau)}\right), \tag{A11}$$

when II is the elliptic integral of the third kind. To evaluate J, make the further change of integration variable $z = \frac{u-\beta}{\alpha-\beta}$, so that (A7) becomes

$$J_1 = \frac{\sqrt{(r-\beta)^3}}{\sqrt{c-\beta}} \int_0^1 \frac{\sqrt{(1-\delta z)^3} \; dz}{\sqrt{z(1-z)(1-rz)}} \quad , \tag{A12}$$

in which $\delta = \frac{\alpha-\beta}{r-\beta}$, and $\gamma = \frac{\alpha-\beta}{c-\beta}$. Formula (A12), as given in tables [3], may be expressed as

$$J_1 = \frac{\sqrt{[r^*(\tau)]^3}}{\sqrt{a+r^*(\tau)}} \; \pi F_1\left(\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}, 1; \; \frac{r^*(\tau)-a}{r^*(\tau)} \; , \; \frac{r^*(\tau)-a}{r^*(\tau)+a}\right) \quad , \tag{A13}$$

where F_1 is a hypergeometric function. Finally, integral J_2 is evaluated by writing

$$J_2 = r \, J_{21} - J_{22} \quad . \tag{A14}$$

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[3] I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

In equation (A14)

$$J_{21} = r\, I_2 - I_1 \quad , \tag{A15}$$

in which I_2 and I_1 are given by (A10) and (A11), and

$$J_{22} = \int_{\beta}^{\alpha} \frac{\sqrt{r-u} \quad du}{\sqrt{(\alpha-u)(c-u)(u-\beta)}} \quad . \tag{A16}$$

The integral J_{22} is given directly by tables [3]; the result in terms of the original variables is

$$J_{22} = \frac{\sqrt{2a}}{\sqrt{r^*(\tau)}} \; \Pi\left(\pi/2, \frac{r^*(\tau)-a}{r^*(\tau)} \; , \frac{\sqrt{r^*(\tau)-a}}{\sqrt{2r^*(\tau)}}\right) \; . \tag{A17}$$

Substituting (A9)-(A17) into (A2)-(A4) gives the result for (A1).

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3 I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

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